An advanced procedure for hydroelastic analysis of very large floating airport exposed to airplane load

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Abstract. Transient response of a pontoon type VLFS subjected to moving load due to airplane landing/take-off is investigated. An approximate solution of free plate natural vibrations, based on a new set of mathematical modes consisting of ordinary products of beam modes and their combinations as additional (extraordinary) modes, for which the natural frequencies are analytically determined using the Rayleigh's quotient, is proposed. The system of modal hydroelastic equations, governing the floating airport forced vibrations, is derived using the Lagrange equation of the second kind and a simplified solution procedure is outlined. Airplane excitation loads are discussed and a new formulation of moving load is proposed and validated. The obtained results are compared with known ones from relevant literature.

Key words: *floating airport; airplane load; transient response*

1. Introduction

Very Large Floating Structures (VLFS) have been actively researched since the 1970's and the field is very well established, [1]. Since length and breadth of a VLFS are significantly larger than the draft, such structure can be modelled as an elastic isotropic floating thin plate with free edges. Because of its size, floating plate responses are dominated by the elastic modes and hydroelasticity becomes an issue with structure responding like a thin sheet floating on water. In [2] it is recognized that solution of the equations of motion for a thin floating plate, within the modal superposition method, can be categorized into two classes of methods: i. the eigenfunction expansion method, where the modes used are usually numerically calculated by the finite element technique for free plate (*physical modes*) and ii. the methods where the response is expanded in modes other than the eigenfunctions of the thin plate (*mathematical modes*). In the latter case those modes are often products of free-free beam modes or 2D polynomial functions, [1] and [2], respectively.

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2. Formulation of the problem

A shallow-draft pontoon-like floating airport of length L and breadth B is considered in the Cartesian coordinate system. The rectangular pontoon is assumed to be double symmetric, and the origin of the coordinate system is placed in the middle of the pontoon due to reason of simplicity, Fig. 1.

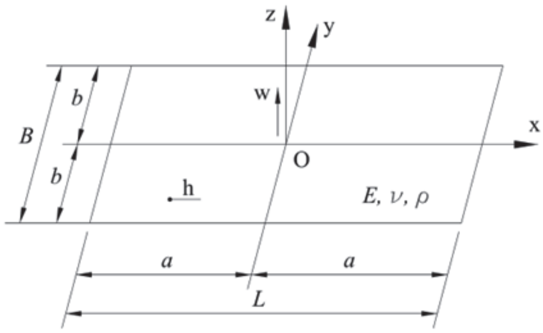

Figure 1. Rectangular pontoon-type VLFS

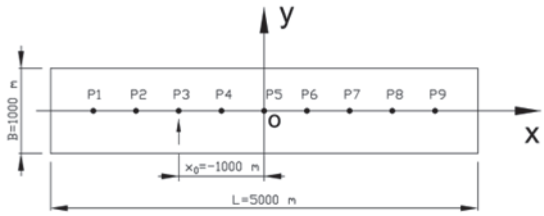

Figure 2. Pontoon dimensions and the calculation points

The pontoon in water is treated as a free thin rectangular plate on elastic support. Differential equation of vibrations is presented in the form

$$D\Delta\Delta w+m\frac{\partial^2 w}{\partial t^2}+\rho gw=p_h-p_e, \tag{1}$$

where $w(x,y,t)$ is plate deflection, $D=Eh^3/[12(1-\nu^2)]$ is plate flexural rigidity (E – Young's modulus, ν - Poisson's ratio, h – equivalent thickness of the pontoon thin-walled structure), $m(x,y)$ is mass per unit area, $\Delta(.)=\partial^2(.)/\partial x^2+\partial^2(.)/\partial y^2$ is two dimensional Laplace differential operator, $\rho g w$ is hydrostatic pressure, $p_h(x,y,t)$ is hydrodynamic pressure, and $p_e\left(x(t),y(t),t\right)$ is local time-dependent moving pressure due to landing and take-off of an airplane. Negative sign of p_e in (1) is related to landing load acting vertically downward. Boundary conditions for a free-free plate read: $\frac{\partial^2 w}{\partial n^2}+\nu\frac{\partial^2 w}{\partial s^2}=0$ and $\frac{\partial^3 w}{\partial n^3}+(2-\nu)\frac{\partial^3 w}{\partial n\partial s^2}=0$.

Differential equation of motion (1) cannot be solved analytically due to local moving load and hydrodynamic forces. Therefore, the energy approach is preferable, by utilizing the modal superposition method. Generally, the forced response of a plate is assumed as a series of products of the free plate natural modes and corresponding time functions

$$w(x,y,t)=\sum_{j=1}^{\infty}W_j(x,y)T_j(t). \tag{2}$$

However, in case of a free rectangular plate it is not possible to determine natural modes in an analytical closed form due to complex boundary conditions. Therefore, so called mathematical modes are used. Usually, the free beam natural modes are applied for this purpose, $X_m(x)$ and $Y_n(y)$ for x and y direction, respectively.

3. Natural modes of free beam

Due to practical reasons it is suitable to distinguish symmetric and antisymmetric modes in domain $-a \le x \le a$

$$
\begin{aligned}
X_0 &= 1, \quad X_m = \frac{1}{2}\left(\frac{\cosh \alpha_m x}{\cosh \alpha_m a} + \frac{\cos \alpha_m x}{\cos \alpha_m a}\right), \quad m = 2,4...\\
X_1 &= \frac{x}{a}, \quad X_m = \frac{1}{2}\left(\frac{\sinh \alpha_m x}{\sinh \alpha_m a} + \frac{\sin \alpha_m x}{\sin \alpha_m a}\right), \quad m = 3,5...
\end{aligned} \tag{3}
$$

where values of α_m are obtained from the corresponding frequency equation for given boundary conditions

$$
\begin{aligned}
&\tanh \alpha a \pm \tan \alpha a = 0,\ \alpha_0 a = 0,\ \alpha_1 a = 0,\ \alpha_2 a = 2.3650,\ \alpha_3 a = 3.9266,\\
&\alpha_4 a = 5.4978,\ \alpha_5 a = 7.0686,\ \lim_{m\to\infty} \alpha_m a = \frac{1}{4}(2m-1)\pi,\ m = 2,3,4...
\end{aligned} \tag{4}
$$

In a similar way $Y(y)$ functions for y direction, where $-b \le y \le b$, can be constructed with parameter β_m.

4. Rigorous solution of free plate natural vibrations

Differential equation of dry plate natural vibrations includes the first two terms of Eq. (1). Natural vibrations are harmonic and one can write

$$w(x,y,t) = W(x,y)\mathrm{e}^{i\omega t}, \tag{5}$$

where W is mode function and ω natural frequency. By substituting (5) into reduced Eq. (1), a modal differential equation is obtained

$$D\Delta\Delta W - \omega^2 m W = 0. \tag{6}$$

Eq. (6) can be solved analytically, but the boundary conditions for the free plate edges cannot be satisfied. Therefore, a numerical method has to be applied.

Usually, the Rayleigh-Ritz method is used, for which purpose it is necessary to specify potential (strain) and kinetic energy of the vibrating system. The plate strain energy, as integral of work of the bending and torsional moments over the plate area, can be presented in a suitable form for further consideration, [3]

$$U = \frac{D}{2}\int_{-b}^{b}\int_{-a}^{a}\left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y. \tag{7}$$

The kinetic energy reads

$$K = \frac{1}{2}\int_{-b}^{b}\int_{-a}^{a} m(\dot{w})^2\,\mathrm{d}x\,\mathrm{d}y. \tag{8}$$

By substituting (5) into (7) expression for modal strain energy is obtained

$$U = \frac{D}{2}\int_{-b}^{b}\int_{-a}^{a}\left[\left(\frac{\partial^2 W}{\partial x^2}\right)^2 + \left(\frac{\partial^2 W}{\partial y^2}\right)^2 + 2\nu\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2} + 2(1-\nu)\left(\frac{\partial^2 W}{\partial x\,\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y. \tag{9}$$

In similar way the modal kinetic energy for uniform mass distribution, reads

$$K = \frac{1}{2}\omega^2 m \int_{-b}^{b}\int_{-a}^{a} W^2 \mathrm{d}x\,\mathrm{d}y . \tag{10}$$

The plate natural mode can be assumed in the form of a series

$$W(x,y) = \sum_{j=1}^{\infty} C_j Z_j(x,y) , \tag{11}$$

where C_j are unknown constants and $Z_j(x,y)$ are appropriate deflection functions, which individually have to satisfy at least the geometric boundary conditions. Functions $Z_j(x,y)$ are usually called the mathematical natural modes, while $W(x,y)$ are physical natural modes.

The modal strain energy has to be equal to the kinetic energy. For approximated natural modes difference has to be minimal. By substituting (11) into (9) and (10) and satisfying condition

$$\frac{\partial U}{\partial C_i} - \frac{\partial K}{\partial C_i} = 0 , \tag{12}$$

one arrives at the system of homogenous algebraic equations

$$\left(\left[S_{ij}^0\right] - \omega^2\left[M_{ij}^0\right]\right)\left\{C_j\right\} = \{0\} , \tag{13}$$

where

$$\begin{aligned} S_{ij}^0 = D\int_{-b}^{b}\int_{-a}^{a} & \left[\frac{\partial^2 Z_i}{\partial x^2}\frac{\partial^2 Z_j}{\partial x^2} + \frac{\partial^2 Z_i}{\partial y^2}\frac{\partial^2 Z_j}{\partial y^2} + \nu\left(\frac{\partial^2 Z_i}{\partial x^2}\frac{\partial^2 Z_j}{\partial y^2} + \frac{\partial^2 Z_i}{\partial y^2}\frac{\partial^2 Z_j}{\partial x^2}\right)\right. \\ & \left. +2(1-\nu)\frac{\partial^2 Z_i}{\partial x\,\partial y}\frac{\partial^2 Z_j}{\partial x\,\partial y}\right]\mathrm{d}x\,\mathrm{d}y, \end{aligned} \tag{14}$$

$$M_{ij}^0 = m\int_{-b}^{b}\int_{-a}^{a} Z_i Z_j \mathrm{d}x\,\mathrm{d}y . \tag{15}$$

are elements of the modal stiffness and mass matrix, respectively. Once spectrum of natural frequencies $\omega_p,\ p = 1,2...,$ for chosen number of terms in the series (11), is determined from the condition

$$\mathrm{Det}\left(\left[S_{ij}^0\right] - \omega\left[M_{ij}^0\right]\right) = 0, \tag{16}$$

the corresponding vector of constants C_{pj} for each ω_p is obtained from (13) by assuming unit value for one of the constants. An appropriate choice is to take $C_{pp} = 1$, since it is expected to be dominant.

The mathematical modes for a free plate in (11) are usually assumed in the form of product of free beam natural modes, Eq. (3).

$$Z_i(x,y) = \left[X_m(x)Y_n(y)\right]_i , \quad Z_j(x,y) = \left[X_k(x)Y_l(y)\right]_j , \tag{17}$$

where $m, n = 1, 2... M$ and $k, l = 1, 2... N$. The index i and j are prescribed to each combination of indices m and n, and k and l, respectively. By substituting (17) into (14) yields

$$S_{ij}^{0} = D\int_{-b}^{b}\int_{-a}^{a}\Big[\left(X_m'' X_k'' Y_n Y_l \delta_{nl}\right)_{ij} + \left(X_m X_k Y_n'' Y_l'' \delta_{mk}\right)_{ij}$$
$$+\nu\left(X_m'' X_k Y_n Y_l'' + X_m X_k'' Y_n'' Y_l\right)_{ij} + 2(1-\nu)\left(X_m' X_k' Y_n' Y_l'\right)_{ij}\Big]\mathrm{d}x\,\mathrm{d}y. \tag{18}$$

The beam modes are orthogonal and that is indicated in (18) by the Kronecker symbol. If $l = n$ than $k = m$ and the first two terms in (18) are diagonal

$$f_1(x,y)_{ij} + f_2(x,y)_{ij} = \Big[\left(X_m''\right)^2 Y_n^2 + X_m^2\left(Y_n''\right)^2\Big]\delta_{ij}. \tag{19}$$

All integrals in (19) per x and y can be solved analytically. For the first two integrals one obtains

$$(I_1 + I_2)_{ij} = \frac{ab}{4}\left(\alpha_m^4 + \beta_n^4\right)\delta_{ij}. \tag{20}$$

The third and fourth terms in (18) generate full matrix.

By substituting formulae (17) for the mathematical modes into (15), the mass matrix reads

$$M_{ij}^{0} = m\int_{-b}^{b}\int_{-a}^{a}\left[X_m X_k Y_n Y_l\right]_{ij}\mathrm{d}x\,\mathrm{d}y. \tag{21}$$

Due to orthogonality of the beam natural modes, the mass matrix is diagonal and after integration yields

$$M_{ij}^{0} = m\int_{-b}^{b}\int_{-a}^{a} X_m^2 Y_n^2\,\mathrm{d}x\,\mathrm{d}y\,\delta_{ij},\quad M_{ij}^{0} = \frac{1}{4}mab\,\delta_{ij}. \tag{22}$$

Hence, elements of diagonal mass matrix are the same for all modes.

5. Approximate solution of free plate natural vibrations

Procedure for rigorous solution of free plate natural vibrations presented in Section 4, is rather complicated especially if one looks to achieve simple analytical solution. Therefore, the procedure can be simplified in such a way that some terms influencing weakly plate response are ignored.

A physical natural mode $W_p(x,y)$ is assumed in the form of a series of mathematical modes $Z_j(x,y)$ with weighting constants C_{pj}, Eq. (11). If mathematical mode $Z_p(x,y)$ is identical to the physical mode $W_p(x,y)$ than constant $C_{pp} = 1$, while $C_{pj} = 0$ for $j \neq p$. A very good approximation of the physical mode by mathematical one results with $0 < \left(C_{pj}\right) \ll 1$ for $j \neq p$. That just happens in the considered case, where the plate natural modes are expressed in the terms of beam natural modes. As a result the first two terms in (18), which represent plate bending as a grillage, according to (20) are diagonal and dominant. Also, since values of constants C_{pi} and C_{pj} for $i, j \neq p$ are very small one can conclude that diagonal elements S_{ii} in (18) are predominant.

By taking the above fact into account the stiffness matrix (18) is reduced to the diagonal elements

$$S_{ij}^0 = D\frac{ab}{4}(\alpha_m^4+\beta_n^4)+D\int_{-b}^{b}\int_{-a}^{a}\left[2\nu X_m'' X_m Y_n'' Y_n + 2(1-\nu)\left(X_m'\right)^2\left(Y_n'\right)^2\right]\mathrm{d}x\,\mathrm{d}y\ . \tag{23}$$

Integrals in (23) can be solved analytically. For double symmetric modes the approximate modal stiffness matrix reads

$$S_{ii}^0 = \frac{D}{4ab}u_{mn}\ , \tag{24}$$

where

$$\begin{aligned} u_{mn} &= \left(\alpha_m^4+\beta_n^4\right)a^2b^2 + 2\nu\alpha_m a\,\beta_n b\tanh\alpha_m a\,\tanh\beta_n b(1-\alpha_m a\,\tanh\alpha_m a)(1-\beta_n b\tanh\beta_n b) \\ &\quad +2(1-\nu)\,\alpha_m a\,\beta_n b\tanh\alpha_m a\,\tanh\beta_n b(3+\alpha_m a\,\tanh\alpha_m a)(3+\beta_n b\tanh\beta_n b). \end{aligned} \tag{25}$$

For an antisymmetric beam mode the hyperbolic function tanh (.) in (25) has to be exchanged with coth (.).

The mass matrix remains the same as in the rigorous solution, Eq. 22. By substituting modal stiffness and modal mass, Eqs. (25) and (22), respectively, into (13) expression for natural frequency of modes $W_{mn} \approx Z_{mn} = X_m Y_n$ is obtained

$$\omega_{mn}^0 = \frac{1}{ab}\sqrt{\frac{D}{m}}\sqrt{u_{mn}}\ , \tag{26}$$

where u_{mn} is defined by (25). Eq. (26) actually represents the Rayleigh's quotient, which application is extended here to higher modes.

Recently performed detailed FEM natural vibration analysis of thin square and rectangular plate shows that natural modes $W_{mn} \approx Z_{mn} = X_m Y_n$ are only one family of possible modes shapes, [4]. Two additional mode families are identified of the type

$$W_{mn\pm kl} \approx Z_{mn\pm kl} = X_m Y_n \pm X_k Y_l\,. \tag{27}$$

Determination of plate strain energy, i.e. natural frequencies, is rather difficult task in the above case as can be seen in [4]. Spectra of natural frequencies of these extraordinary modes are mixed with that of the ordinary modes for a square plate. For a rectangular plate the extraordinary frequency spectra are shifted to the higher modes according to increasing value of the plate aspect ratio *a/b*.

6. Rigorous solution of floating airport forced vibrations

Problem is analysed by the energy approach and modal superposition method. Total energy of the vibrating airport consists of strain energy, U , kinetic energy, K, energy of hydrostatic pressure as restoring force, U_R, and energy of hydrodynamic forces, E_h, which has to be equal to the work of external load, V. The hydrodynamic forces require a special treatment due to complexity of hydrodynamic load, and are considered separately in the next Section and are taken later on into account.

Hence, one can write for the time being

$$E_{\mathrm{tot}} = U + K + U_R = V\ , \tag{28}$$

where U and K are specified by (7) and (8), respectively and

$$U_R = \frac{1}{2}\rho g\int_{-b}^{b}\int_{-a}^{a} w^2\,\mathrm{d}x\,\mathrm{d}y,\quad V = -\int_{-b}^{b}\int_{-a}^{a} p_e\, w\,\mathrm{d}x\,\mathrm{d}y\,. \tag{29}$$

The pontoon forced deflection w (x, y, t) is assumed in the form of a series of products of natural modes $W_j(x,y)$ and the unknown time functions $T_j(t)$, Eq. (2).

The pontoon total energy (28) expressed in the terms of natural modes (2), has to satisfy the Lagrange equation of the second kind

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial K}{\partial \dot{T}_i}\right)-\frac{\partial K}{\partial T_i}+\frac{\partial (U+U_R)}{\partial T_i}=\frac{\partial V}{\partial T_i}\,. \tag{30}$$

By employing equations (7), (8) and (29) for the energy components, one arrives at the system of the uncoupled modal equations as a result of the orthogonality of natural modes

$$M_i\ddot{T}_i+(S_i+R_i)T_i=F_i(t), \quad i=1,2\,... \tag{31}$$

where according to (9), (10) and (29)

$$S_i=D\int\limits_{-b}^{b}\int\limits_{-a}^{a}\left[\left(\frac{\partial^2 W_i}{\partial x^2}\right)^2+\left(\frac{\partial^2 W_i}{\partial y^2}\right)^2+2\nu\frac{\partial^2 W_i}{\partial x^2}\frac{\partial^2 W_i}{\partial y^2}+2(1-\nu)\left(\frac{\partial^2 W_i}{\partial x\,\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y \tag{32}$$

$$M_i=m\,I_i,\; R_i=\rho\,g\,I_i,\; I_i=\int\limits_{-b}^{b}\int\limits_{-a}^{a}W_i^2\,\mathrm{d}x\,\mathrm{d}y,\; F_i(t)=-\int\limits_{-b}^{b}\int\limits_{-a}^{a}p_e\left(x(t),y(t),t\right)W_i\,\mathrm{d}x\,\mathrm{d}y \tag{33}$$

The above parameters are modal stiffness, mass, restoring stiffness and excitation force, respectively. It is obvious that parameters M_i and R_i are of the same structure.

System of the pontoon modal equations has to be extended with the hydrodynamic inertia and damping force, as well as structural damping force

$$\sum_{j=1}^{\infty}\left\{A_{ij}(\infty)\ddot{T}_j+\int\limits_0^t K_{ij}(t-\tau)\dot{T}_j(\tau)\mathrm{d}\tau\right\}+M_i\ddot{T}_i+D_i\dot{T}_i+(S_i+R_i)T_i=F_i(t),\; i=1,2\,...\,, \tag{34}$$

where D_i is damping coefficient, $A_{ij}(\infty)$ is infinite frequency added mass, and

$$K_{ij}(t)=\frac{2}{\pi}\int\limits_0^{\infty}B_{ij}(\omega)\cos\omega t\,\mathrm{d}\omega \tag{35}$$

is the frequency dependent part of the wave radiation force. Matrices $A_{ij}(\infty)$ and $K_{ij}(t)$ are full matrices, and consequently the modal equations are coupled. Integral in (34) is the impulse response (memory) function, which can be calculated from the linear frequency dependent damping coefficient (35).

Structural damping coefficient can be specified in an usual form $D_i=2\omega_i\xi_i M_i$, where ξ_i is dimensionless damping coefficient given as a percentage of the critical value $D_{i\,cr}=2\sqrt{M_i S_i}$.

The pontoon modal vibration parameters S_i, M_i and R_i expressed in terms of physical natural modes can be expanded into terms of mathematical modes. For that purpose expressions (11) and (17) can be written in matrix notation

$$W_i=\left\langle C_{ip}\right\rangle\left\{Z_p\right\}, \quad Z_p=(X_m Y_n)_p, \quad Z_q=(X_k Y_l)_q\,. \tag{36}$$

One should note $\langle\cdot\rangle$ and $\{\cdot\}$ represent row and column vectors, respectively. By substituting (36) into (32) and (33) and using orthogonality conditions for the beam natural modes, one arrives at

$$S_i = D\langle C_{ip}\rangle \int_{-b}^{b}\int_{-a}^{a}\left\{\left[\left((X_m'')^2\, Y_n^2\right)_p\right]+\left[\left(X_m^2\,(Y_n'')^2\right)_p\right]\right\}\mathrm{d}x\,\mathrm{d}y\{C_{ip}\}$$
$$+2D\langle C_{ip}\rangle \int_{-b}^{b}\int_{-a}^{a}\left\{\nu\left[\left(X_m''\,X_k\,Y_n\,Y_l''\right)_{pq}\right]+(1-\nu)\left[\left(X_m'\,X_k'\,Y_n'\,Y_l'\right)_{pq}\right]\right\}\mathrm{d}x\,\mathrm{d}y\,\{C_{ip}\} \quad (37)$$

$$M_i = mJ_i,\; R_i = \rho g J_i,\; J_i = \langle C_{ip}\rangle \int_{-b}^{b}\int_{-a}^{a}\left\{\left[\left(X_m^2\, Y_n^2\right)_p\right]\right\}\mathrm{d}x\,\mathrm{d}y\{C_{ip}\}, \quad (38)$$

The orthogonality conditions are used in the first two terms of (37) as well as in (38). Those matrices are diagonal, while the last two matrices in (37) are full.

Fortunately, it is not necessary to calculate rather complex modal stiffness S_i since, once the natural vibration calculation is performed according to Section 4, one can write $S_i = \omega_i^2 M_i$. Also, the modal restoring stiffness R_i is of the same structure as M_i, Eq. (38). As a result only modal mass M_i has to be determined. Integral in (38) takes value $ab/4$ and is common for all modes. Hence, one obtains

$$M_i = \frac{1}{4} mab \sum_{p=1}^{\infty} C_{ip}^2 . \quad (39)$$

7. Simplified solutions of floating airport forced vibrations

The rigorous procedure for airport forced vibration analysis, presented in the previous Section, can be simplified by avoiding the natural vibration analysis as a prerogative. That is achieved by assuming airport pontoon deflection directly in the form of a series of product of mathematical modes and time functions, as it is usually done in the literature, [5]

$$w(x,y,t) = \sum_{j=1}^{\infty} Z_j(x,y) T_j(t) . \quad (40)$$

In this way stiffness matrix and mass matrix are identical to those in natural vibration analysis, S_{ij}^0, Eqs. (14) and (18) and M_{ij}^0, Eqs. (15) and (22). The restoring stiffness matrix, based on the similarity with M_{ij}^0, is $R_{ij}^0 = \rho g M_{ij}^0 / m$.

The infinite frequency added mass matrix, the hydrodynamic damping matrix in (34), and the excitation force (33), are determined for the mathematical mode $Z_i = (X_m Y_n)_i$. Analytical mathematical modes represent some advantage for calculation of the hydrodynamic forces.

Furthermore, as it is already explained in Section 5 related to natural vibrations, if a physical natural mode, $W_i(x,y)$, is very well approximated by mathematical one, $Z_i(x,y)$, then choosing constant $C_{ii} = 1$, the other constants are very small, i.e. $0 < C_{ij} \ll 1$, for $j \neq i$. By taking this fact into account, Eqs. (37) and (38) are reduced to the approximate expressions: S_{ij}^0, Eq. (24), M_{ij}^0, Eq. (22) and $R_{ij}^0 = \rho g M_{ij}^0 / m$. All these parameters are expressed analytically, while the hydrodynamic coefficients have to be determined numerically, Eqs. (34) and (35).

Influence of the hydrodynamic damping force in (34) on transient response is quite small and therefore can be omitted. If only diagonal elements of the added mass matrix are taken into account, the modal differential equations are decomposed. A typical modal equation can be presented in the form

$$\left(mab+4A_{ii}(\infty)\right)\ddot{T}_i+2\omega_i\xi_i mab\dot{T}_i+\left(\omega_i^2 m+\rho g\right)abT_i=4F_i(t),\quad i=1,2\ldots,\qquad(41)$$

Solution of Eq. (41) can be used for initial estimation of the airport transient response.

After an airplane landing and take-off the airport pontoon continues to vibrate in a natural manner, since there is no external excitation. Decreasing of the response depends on structural and hydrodynamic damping intensity.

8. Hydrodynamic coefficients

Using an appropriate Green function, implicitly satisfying the boundary conditions at the free surface and the radiation condition, linear water-wave problem (namely the radiation problem) is solved using the Boundary Integral Equation method (in the form of source distributions), which in the discretized form (constant panel method) reads

$$\frac{1}{2}\sigma(P_i)+\sum_{j=1}^{N}\sigma(P_j)\cdot\frac{1}{4\pi}\iint_{s_j}\frac{\partial}{\partial n}G(P_i,Q)\,\mathrm{d}S(Q)=f_p(P_i),\quad i=1,2\ldots,N,\qquad(42)$$

where σ is the unknown source strength, $G(P,Q)$ is the free surface Green function, [6], $f_p(P_i)=\vec{Z}_j\cdot\vec{n}$ is the generalized radiation mode projected onto corresponding panel unit normal, N is the total number of panels discretizing the wetted surface of the body and S_j is the surface of the corresponding panel. After the unknown source strengths have been solved velocity potential for the radiation problem is calculated using

$$\phi(P)=\sum_{j=1}^{N}\sigma_j\cdot\frac{1}{4\pi}\iint_{S_j}G(P,Q)\,\mathrm{d}S(Q).\qquad(43)$$

After the radiation potentials have been calculated modal added mass and damping read

$$A_{ij}(\omega)=\mathrm{Re}\left\{\rho\int_{S_B}\phi_j\,\vec{Z}_i\cdot\vec{n}\,\mathrm{d}S\right\},\quad B_{ij}(\omega)=\mathrm{Im}\left\{\omega\rho\int_{S_B}\phi_j\,\vec{Z}_i\cdot\vec{n}\,\mathrm{d}S\right\},\qquad(44)$$

where ϕ_j is the i-th mode radiation potential and S_B is the wetted surface of the pontoon. It is recognized by the authors that there are more efficient methods available in order to calculate the wave radiation forces, [7], and we are currently working on implementing higher order BEM method for the VLFS problems.

9. Airplane excitation of floating airport vibrations

An airplane on land is usually supported in three points by the landing gears: the front one below the navigation cabin and another two below the wings. Even in the case of a jumbo jet with more supports their spans are relatively small compared to the pontoon dimensions. Therefore, the airplane load during landing and take-off is simulated as a moving time – varying lumped force below the airplane centre of gravity.

It is assumed that an airplane moves along the runway during landing and take-off uniformly accelerated, with constant acceleration a_0. Hence, the airplane velocity and position read

$$v(t) = v_0 + a_0 t, \quad x_p(t) = x_0 + v_0 t + \frac{1}{2} a_0 t^2 , \tag{45}$$

where v_0 and x_0 are the airplane initial velocity and position, respectively.

The airplane lift force is calculated by the formula, [5]

$$F_L(t) = \frac{1}{2} \rho_a A_w v^2(t) C_L(t), \quad C_L(t) = a_L \,\mathrm{e}^{b_L t} , \tag{46}$$

where ρ_a is air density, A_W is effective wing area, $C_L(t)$ is coefficient of lift force with constant but different values of parameters a_L and b_L for landing and take-off, [5].

The total force acting on a plane is difference between its weight, Q, and lift force

$$F(t) = Q - F_L(t) . \tag{47}$$

Take-off. In this operation the airplane initial velocity $v_0 = 0$, and according to (45)

$$v(t) = a_0 t, \quad x_p(t) = x_0 + \frac{1}{2} a_0 t^2 . \tag{48}$$

Necessary time for take-off, t_{t0}, is obtained from condition $F(t) = 0$, i.e.

$$Q - \frac{1}{2} \rho_a A_w a_0^2 a_L t^2 \,\mathrm{e}^{b_L t} = 0 . \tag{49}$$

In that time instant $v_{t0} = a_0 t_{t0}$ and $x_{p_{t0}} = x_0 + \frac{1}{2} a_0 t_{t0}^2$. Total force during take-off is

$$F_{t0}(t) = Q - \frac{1}{2} \rho_a A_w a_0^2 a_L t^2 \,\mathrm{e}^{b_L t} , \tag{50}$$

Landing. This operation starts when $F(t) = 0$. The initial velocity v_0 at $t = 0$ is determined from condition

$$Q - \frac{1}{2} \rho_a A_w v_0^2 a_L = 0 . \tag{51}$$

The initial airplane position x_0 is known. The airplane velocity and position are determined by Eqs. (45), while a_0 is retardation. The total force during landing is

$$F_l(t) = Q - \frac{1}{2} \rho_a A_w (v_0 + a_0 t)^2 a_L \,\mathrm{e}^{b_L t} . \tag{52}$$

In the modal superposition method the modal excitation load is specified by (33). Since integration of local pressure per pontoon area results in lumped total force $F(t)$, Eq. (47), its work, which represents modal force, reads

$$F_i(t) = -F(t)\, Z_i\left(x(t), y(t)\right), \tag{53}$$

where the modal index *i* corresponds to one combination of the indices *m* and *n*. It is assumed that an airplane moves along the pontoon longitudinal centreline. Therefore, only symmetric beam natural modes are involved in the analysis, $Y_n(y)$, *n* = 0, 2, 4... The coordinate *x* has to be expressed by the time-varying airplane position coordinate $x_p(t)$ for landing and take-off, Eqs. (45) and (48), respectively.

10. Illustrative Example

10.1. Numerical Calculations

A benchmark realistic example of landing and take-off of a Boeing 747-400 airplane on/of rectangular floating airport as a VLFS is considered. The basic airport and airplane data are taken from [5]. The pontoon dimensions and the airplane starting point in landing and take-off, together with nine points for registration of the elastic deflection, are shown in Fig. 2. The time histories of the airplane position $x_p(t)$ and velocity $v(t)$, Eqs. (45) and (48), and the excitation forces $F_l(t)$ and $F_{t0}(t)$, Eqs. (52) and (50), for landing and take-off, respectively, are the same as those presented in [5].

Natural vibrations of the floating airport are determined by the finite element method. Values of the first 20 natural frequencies of transversally symmetric modes are listed in Table 1. The mode shapes are analysed and their type is also indicated in Table 1. All modes are of the type $X_m Y_n$ except mode no. 8 of complex shape $X_8 + Y_2$. Approximate values of natural frequencies determined for analytical modes by the Rayleigh's quotient are given in Table 1. Their discrepancies with respect to the FEM values are acceptable for an approximate estimation. For illustration three typical natural and mathematical modes are compared in Fig. 3.

The first 20 natural frequencies for each type of actual ordinary and extraordinary mathematical modes are listed in Table 2. Only transversely symmetric modes are included due to assumed airplane run along pontoon centreline. Choice of mathematical modes in series of forced response is based on the level of natural frequency. Hence, the forced vibration analysis is performed by taking 20 W_{m0} modes, 18 W_{m2} modes, 9 W_{m4} modes and extraordinary mode W_{8+2} for faster convergence, into account. Governing system of both coupled and uncoupled ordinary differential equations, Eqs. (34) and (41), respectively, is solved using the explicit 4th order Runge-Kutta method and Dormand and Prince coefficients. In order to calculate $A_{ij}(\infty)$ reliably, wetted surface of the pontoon has been discretized into 14300 panels.

Table 1. Frequency parameter $\Omega = \left(\omega L^2 / \pi^2\right)\sqrt{\rho h / D}$ of free pontoon

Mode no.	Mode type	FEM	RQ,PP	Discrepancies %
1	X_2Y_0	2.1671	2.2668	4.60
2	X_3Y_0	6.0020	6.2485	4.11
3	X_4Y_0	11.8235	12.2496	3.60
4	X_5Y_0	19.6265	20.2492	3.17
5	X_6Y_0	29.4075	30.2488	2.86
6	X_7Y_0	41.1249	42.2483	2.73
7	X_8Y_0	54.0880	56.2478	3.99
8	X_8+Y_2	57.0513	57.6414	1.03
9	X_1Y_2	57.7312	58.5916	1.49
10	X_2Y_2	62.9357	64.1839	1.98
11	X_3Y_2	69.1708	72.6235	4.99
12	X_9Y_0	72.2657	72.2472	-0.03
13	X_4Y_2	79.8700	82.8911	3.78
14	$X_{10}Y_0$	89.1405	90.2465	1.24
15	X_5Y_2	91.7001	94.9080	3.50
16	X_6Y_2	105.1316	108.5228	3.22
17	$X_{11}Y_0$	108.7891	110.2457	1.34
18	X_7Y_2	120.1243	123.6939	2.97
19	$X_{12}Y_0$	130.4665	132.2448	1.36
20	X_8Y_2	136.6954	140.4294	2.73

Table 2. Frequency parameter $\Omega = \left(\omega L^2 / \pi^2\right)\sqrt{\rho h / D}$ of free pontoon

Mode type / Mode no.	$W_{m0}=X_mY_0$	$W_{m2}=X_mY_2$	$W_{m+2}=X_m+Y_2$	$W_{m-2}=X_m-Y_2$	$W_{m4}=X_mY_4$
0	0	54.1650			292.7025
1	0	58.5916			307.5557
2	2.2668	64.1839	40.4341	39.7707	311.9253
3	6.2485	72.6235	40.8688	39.7527	319.1178
4	12.2496	82.8911	41.7578	40.2222	328.6107
5	20.2492	94.9080	43.4935	41.5912	340.4688
6	30.2488	108.5228	46.4990	44.3207	354.5744
7	42.2483	123.6939	51.1381	48.7986	370.8341
8	56.2478	140.4294	57.6414	55.2519	389.1627
9	72.2472	158.7637	66.0942	63.7393	409.4876
10	90.2465	178.7418	76.4775	74.2099	431.7493
11	110.2457	200.4110	88.7222	86.5677	455.9008
12	132.2448	223.8163	102.7468	100.7138	481.9070
13	156.2439	248.9981	118.4763	116.5634	509.7430
14	182.2429	275.9920	135.8483	134.0492	539.3924
15	210.2418	304.8282	154.8138	153.1201	570.8460
16	240.2406	335.5323	175.3348	173.7376	604.1000
17	272.2394	368.1260	197.3819	195.8727	639.1549
18	306.2380	402.6272	220.9329	219.5037	676.0147
19	342.2366	439.0507	245.9703	244.6139	714.6854
20	380.2351	477.4091	272.4809	271.1908	755.1750

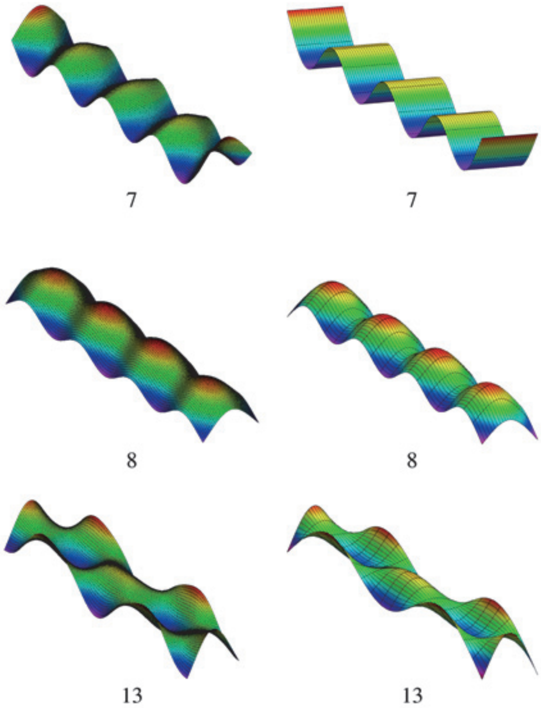

Figure 3. Comparison of FEM and mathematical modes

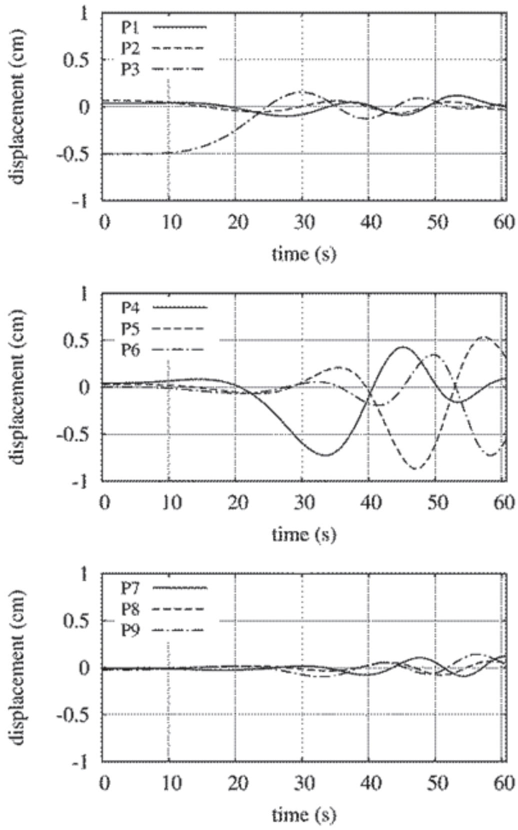

Figure 4. Take-off time history

10.2. Take-off and Landing Analysis

Initial condition of static deflection due to airplane position at P3 is determined by solving Eq. (34), in which the inertial and damping terms are neglected. One should note that the convergence of the modal expansion of the static deflection solution is rather slow (as compared to the direct solution) and if ordinary used dynamic natural modes are employed, it is expected that the initial static solution could be more important than the dynamic considerations when determining the number of required modes in the expansion. Initial velocity is assumed to be zero. Resulting time histories of vertical motions at points P1-P9 are shown in Fig. 4. Resulting perspective view of VLFS vertical deflections at different time instants is given in Fig. 5. Additionally, comparison between the time history of vertical deflection at P3 determined by the present procedure and that presented in [5], is shown in Figs. 6a and 6b. These two figures differ in the sense whether the extraordinary mode (identified from the respective FEM analysis) is included or not in the modal expansion of the solution. It is observed that the difference between the solutions comes from the fact that in the present analysis contribution from the frequency dependent part of the wave radiation force is neglected. Influence of mode coupling (hydrodynamic coupling due to the infinite frequency added mass) on the modal amplitudes is shown in Fig. 7. It can be concluded that this part of coupling, although resulting in visible differences when comparing the respective modal amplitudes, Fig. 7, does not significantly affect the pontoon response, Fig. 6.

For the landing analysis initial conditions of zero static deflection and velocity are assumed. The airplane lands on the floating airport at point P3. Resulting perspective view of VLFS vertical deformations at different time instants is given in Fig. 8. Resulting time histories of vertical motions at points P1-P9 are given in Fig. 9. The same can be concluded, as already stated for take-off analysis. It should be noted that in both landing and take-off analysis it was found that the influence of structural damping was very small in terms of the response amplitudes.

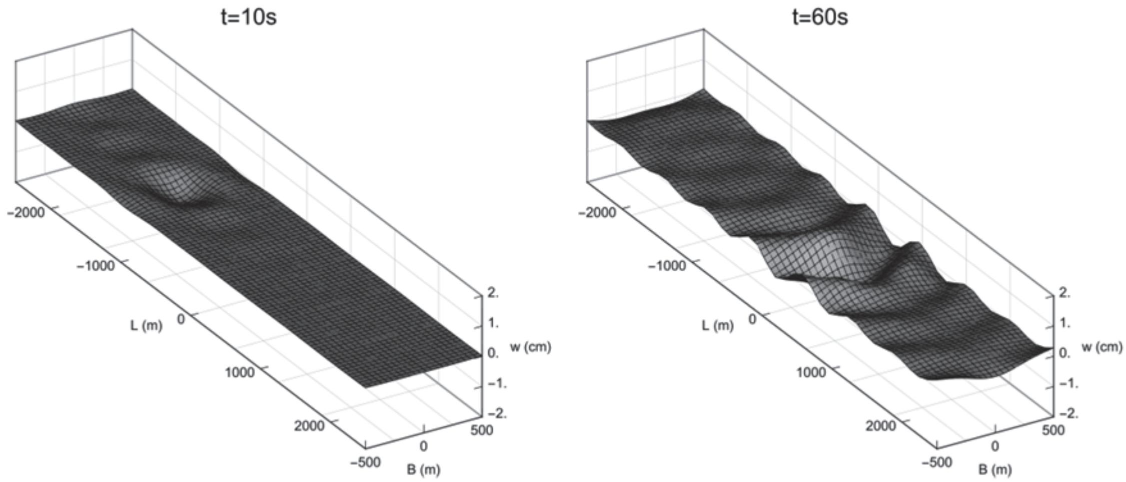

Figure 5. Perspective view of VLFS vertical deformations due to take-off

11. Conclusion

Based on the performed transient vibration analysis of the floating airport the following conclusions can be drawn: (i) Free plate natural vibrations can be solved approximately using a new set of mathematical modes consisting of ordinary products of beam modes and their combinations as additional (extraordinary) modes. Natural frequencies of mathematical modes are analytically calculated using the Rayleigh's quotient; (ii) The system of modal hydroelastic equations can be derived straightforwardly using the Lagrange equation of the second kind; (iii) Rigorous solution of floating airport forced vibrations is derived and simplified and approximate solution procedure is outlined and validated through numerical simulations. Number of chosen mathematical modes is based on their natural frequencies as well as time step in numerical integration. Influence of mode coupling due to hydrodynamics is exemplified. The necessity of inclusion of frequency dependent part of wave radiation forcing is noted; (iv) New formulation of airplane moving load is validated against the known formulation in the literature and it can be concluded that the Gaussian like forcing can be avoided, as can be noted from Fig. 10.

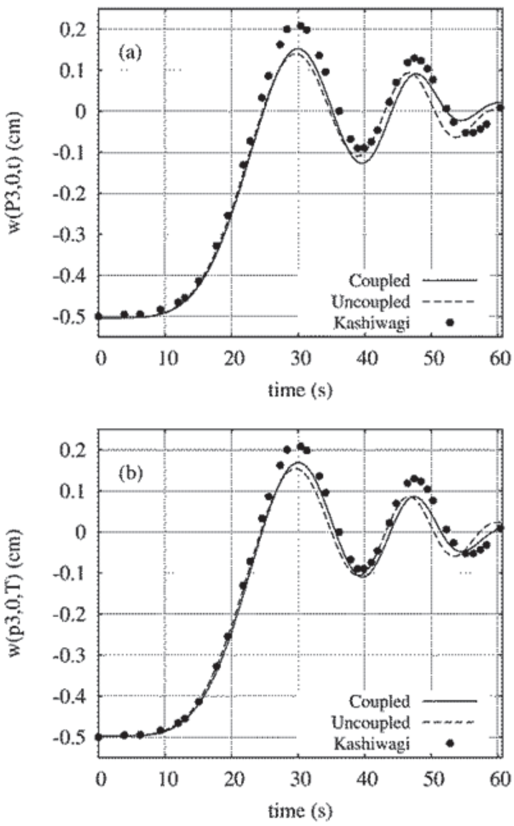

Figure 6. Influence of the extraordinary mode on the response time history

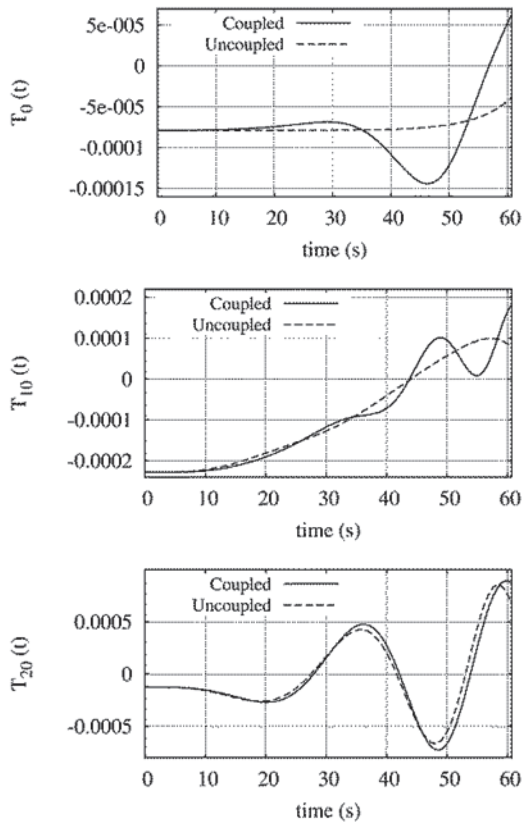

Figure 7. Influence of hydrodynamic coupling due to Aij(∞) on modal amplitudes

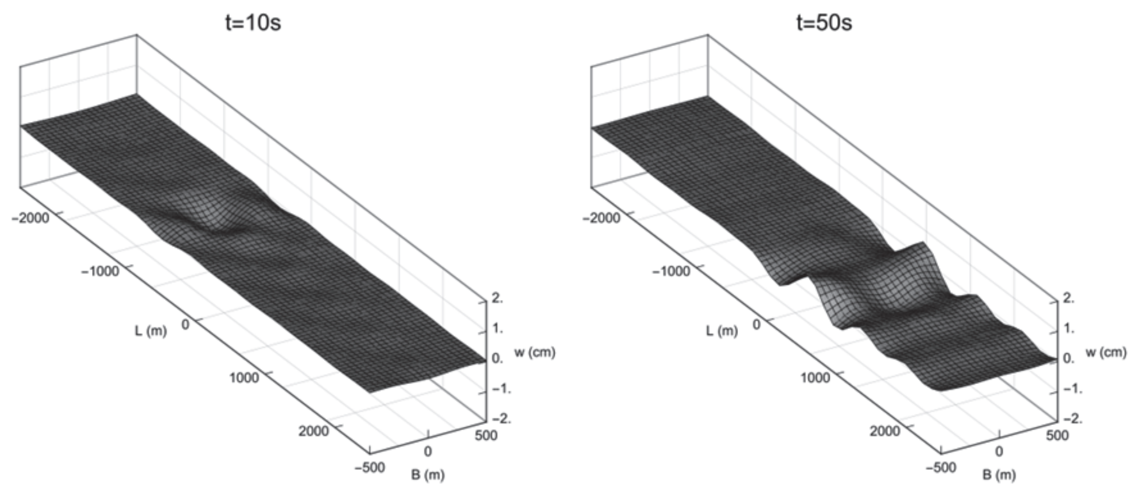

Figure 8. Perspective view of VLFS vertical deformations due to landing

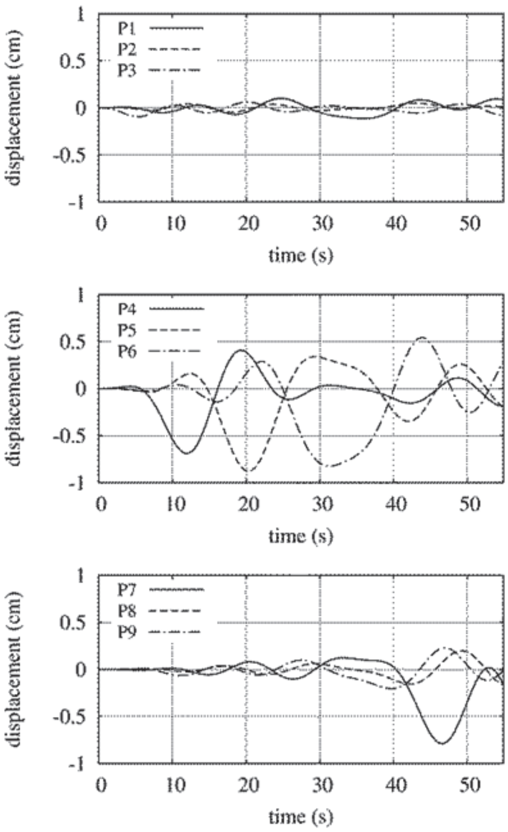

Figure 9. Landing time-history

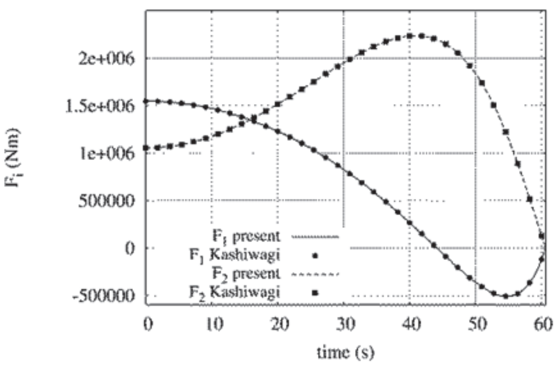

Figure 10. Comparison of different modal forcing formulations

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